Equations of Nonlinear Acoustics and Weak Shock Propagation

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Outline

1) Algorithms Historical

2) Comparative Derivation of the KZK [Zabolotskaya & Khokhlov (1969), Kuznetsov (1971)] **& NPE** [McDonald & Kuperman (1987)] **Equations in a Thermoviscous and Refractive Medium**

3) Comparison of the KZK & NPE Time Domain Algorithms

4) Examples of Numerical Results with the NPE Code

5) Conclusion

1) Algorithms Historical

1) Frequency-domain algorithms [Fenlon (1971), Korpel (1980), Trivett (1981), Pishchal'nikov (1996), ...]

- Constraint : Small absorption → large number of harmonics

2) Time-domain algorithms [McDonald, Kuperman (1987), Lee, Hamilton (1995), Cleveland et al (1996), ...]

- suitable for signals having an arbitrary waveform (pulses).

3) hybride algorithms [Pestorius, Blackstock (1973-74), Bakhalov (1976-87), McKendree (1981), Frøysa (1993), ...]

- nonlinear steepening in the time domain, absorption and diffraction in the frequency domain

2) KZK in a thermoviscous & refractive fluid

Nonlinear wave equation for a refractive medium :

$$\underbrace{\partial_t^2\phi-\left((c_0+c_1)^2+\delta\,\partial_t\right)\Delta\phi}_{Left}=\underbrace{\partial_t\left[(\nabla\phi)^2+\frac{\beta-1}{(c_0+c_1)^2}(\partial_t\phi)^2\right]}_{Right}$$

Slow scale in the direction x of propagation

primes denotes the pulse-following coordinate system

$$\begin{cases}\tau=t-x/c_0\\ x'=\varepsilon\,x\\ y'=\varepsilon^{1/2}\,y\\ z'=\varepsilon^{1/2}\,z\end{cases}\Rightarrow\begin{cases}\partial_t=\partial_\tau\\ \partial_x=\varepsilon\,\partial_{x'}-\dfrac{1}{c_0}\partial_\tau\\ \partial_y=\varepsilon^{1/2}\,\partial_{y'}\\ \partial_z=\varepsilon^{1/2}\,\partial_{z'}\end{cases}\Rightarrow$$

$$\Delta=\partial_x^2+\partial_y^2+\partial_z^2=\varepsilon\,\nabla'^2_\perp+\cancel{\varepsilon^2\partial_{x'}^2}-\varepsilon\frac{2}{c_0}\partial_{x'}\partial_\tau+\frac{1}{c_0^2}\partial_\tau^2$$

$$\Rightarrow\quad Left=2c_0\partial_{x'}\partial_\tau\phi-c_0^2\,\nabla'^2_\perp\,\phi-\frac{\delta}{c_0^2}\partial_\tau^3\phi-\frac{2c_1}{c_0}\partial_\tau^2\phi$$

$$\nabla\approx-\frac{1}{c_0}\partial_\tau\Rightarrow Right=\frac{\beta}{c_0^2}\partial_\tau(\partial_\tau\phi)^2$$

$$\left.\begin{array}{l}v=-\nabla\phi\approx\dfrac{1}{c_0}\partial_\tau\phi\\ v=\dfrac{p'}{\rho_0c_0}\end{array}\right\}\partial_\tau\phi=\frac{p'}{\rho_0}$$

$$\frac{\rho_0}{2c_0}\partial_\tau(Left)=\partial_{x'}\partial_\tau p'-\frac{c_0}{2}\nabla'^2_\perp\,p'-\frac{\delta}{2c_0^3}\partial_\tau^3p'-\frac{c_1}{c_0^2}\partial_\tau^2p'$$

$$\frac{\rho_0}{2c_0}\partial_\tau(Right)=\frac{\beta}{2\rho_0c_0^3}\partial_\tau^2p'^2$$

$$\Rightarrow\quad\frac{\partial^2p'}{\partial x'\partial\tau}-\frac{c_0}{2}\nabla'^2_\perp\,p'-\frac{\delta}{2c_0^3}\frac{\partial^3p'}{\partial\tau^3}-\frac{c_1}{c_0^2}\frac{\partial^2p'}{\partial\tau^2}=\frac{\beta}{2\rho_0c_0^3}\frac{\partial^2p'^2}{\partial\tau^2}$$

KZK equation (including refraction)

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2) NPE in a thermoviscous & refractive fluid

Nonlinear wave equation for a refractive medium :

$$\partial_t^2\phi - \left((c_0+c_1)^2 + \delta\,\partial_t\right)\Delta\phi = \partial_t\left[(\nabla\phi)^2 + \frac{\beta-1}{(c_0+c_1)^2}(\partial_t\phi)^2\right]$$

primes denotes the pulse-following coordinate system

$$\begin{cases} t' = \varepsilon\, t \\ x' = x - c_0\, t \\ y' = \varepsilon^{1/2}\, y \\ z' = \varepsilon^{1/2}\, z \end{cases} \Longrightarrow \begin{cases} \partial_t = \varepsilon\,\partial_{t'} - c_0\,\partial_{x'} \\ \partial_x = \partial_{x'} \\ \partial_y = \varepsilon^{1/2}\,\partial_{y'} \\ \partial_z = \varepsilon^{1/2}\,\partial_{z'} \end{cases} \Longleftrightarrow$$

Lagrangian derivative (moving coordinate system)

$$\partial_t = D_t - c_0\partial_x$$

$$\left(\cancel{D_t^2} + c_0^2\partial_x^2 - 2c_0 D_t\partial_x\right)\phi - \left((c_0+c_1)^2 + \delta\left(\cancel{D_t} - c_0\partial_x\right)\right)\Delta\phi = \left(\cancel{D_t} - c_0\partial_x\right)\left[(\nabla\phi)^2 + \frac{\beta-1}{(c_0+c_1)^2}\left(\left(\cancel{D_t} - c_0\partial_x\right)\phi\right)^2\right]$$

Velocity potential : $\phi = -\int_{-\infty}^{x} v\,dx$

$$\nabla \approx \partial_x \Longrightarrow (\nabla\phi)^2 = {v_x}^2 \approx v^2$$

$$\Longrightarrow 2c_0 D_t v + c_0^2\int_{-\infty}^{x}\nabla_\perp^2 v\,dx + 2c_0c_1\partial_x v - \delta\, c_0\Delta v = -c_0\beta\,\partial_x v^2$$

Linear plane wave impedance relation : $v = p'/\rho_0 c_0$

NPE

[McDonald and Kuperman, *J. Acoust. Soc. Am.* **81**, 1406-1417, (1987)]

$$D_t p' = -\partial_x\left[c_1 p' + \frac{\beta}{2\rho_0 c_0}p'^2\right] - \frac{c_0}{2}\int_{-\infty}^{x}\nabla_\perp^2 p'\,dx + \frac{\delta}{2}\Delta p'$$

2) Similarity between NPE and KZK

KZK equation

[Zabolotskaya and Khokhlov, *Sov. Phys. Acoust.* **15**, 35-40, (1969)]
[Kuznetsov, *Sov. Phys. Acoust.* **16**, 467-470, (1971)]

$$\frac{\partial^2 p}{\partial x_{KZK} \partial t_{KZK}} - \frac{c_0}{2}\nabla_\perp^2 p - \frac{\delta}{2c_0^3}\frac{\partial^3 p}{\partial t_{KZK}^{\;3}} - \frac{c_1}{c_0^2}\frac{\partial^2 p}{\partial t_{KZK}^{\;2}} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial^2 p^2}{\partial t_{KZK}^{\;2}}$$

NPE

[McDonald and Kuperman, *J. Acoust. Soc. Am.* **81**, 1406-1417, (1987)]

$$\partial_{t_{NPE}} p = -\frac{c_0}{2}\int_{-\infty}^{x_{NPE}} \nabla_\perp^2 p \, dx_{NPE} - \partial_{x_{NPE}}\left[c_1 p + \frac{\beta}{2\rho_0 c_0} p^2\right] + \frac{\delta}{2}\partial_{x_{NPE}}^2 p$$

NPE (x_{NPE}, t_{NPE}) ↔KZK (x_{KZK}, t_{KZK})

$$\begin{cases} x_{NPE} = -c_0 t_{KZK} \\ t_{NPE} = t_{KZK} + \dfrac{x_{KZK}}{c_0} \end{cases} \qquad \begin{cases} \partial_{t_{NPE}} = c_0 \partial_{x_{KZK}} \\ \partial_{x_{NPE}} = -\dfrac{1}{c_0}\partial_{t_{KZK}} \end{cases}$$

The roles of propagation distance and time are reversed

KZK equation (including refraction)

$$\partial_{x_{KZK}} p = \frac{c_0}{2}\int_{-\infty}^{t_{KZK}} \nabla_\perp^2 p \, dt_{KZK} + \frac{1}{c_0^2}\partial_{t_{KZK}}\left(c_1 p + \frac{\beta}{2\rho_0 c_0} p^2\right) + \frac{\delta}{2c_0^3}\partial_{t_{KZK}}^2 p$$

3) Time-domain algorithms based on the KZK equation or the NPE

KZK algorithm

[Lee and Hamilton, *J. Acoust. Soc. Am.* **97**, 906-917, (1995)]

$$\partial_x p = \frac{\beta p}{\rho_0 c_0^3}\partial_t p + \frac{c_0}{2}\int_{-\infty}^{t} \nabla_\perp^2 p\, dt + \frac{\delta}{2c_0^3}\partial_t^2 p$$

NPE algorithm

[McDonald and Kuperman, *J. Acoust. Soc. Am.* **81**, 1406-1417, (1987)]

$$\partial_t p = -\partial_x\left[c_1 p + \frac{\beta}{2\rho_0 c_0} p^2\right] - \frac{c_0}{2}\int_{-\infty}^{x} \nabla_\perp^2 p\, dx$$

Each term of the equation is calculated separately over each

spatial incremental step Δx_{KZK}

time incremental step Δt_{NPE} ($\Delta t_{NPE} = \Delta x_{NPE} / c_0$ to get a time waveform at a fixed location)

Integration region (moving time frame) :

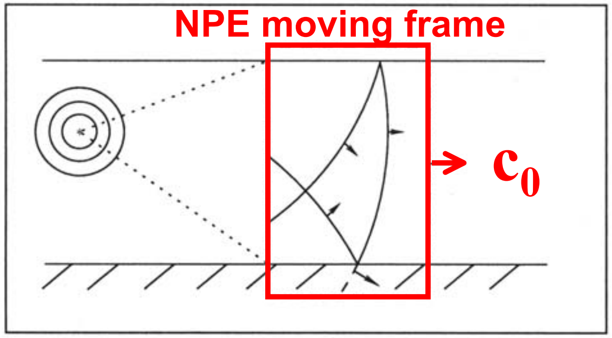
$$t_{\min} \le t_{KZK} \le t_{\max}$$

Integration region (moving spatial frame) :

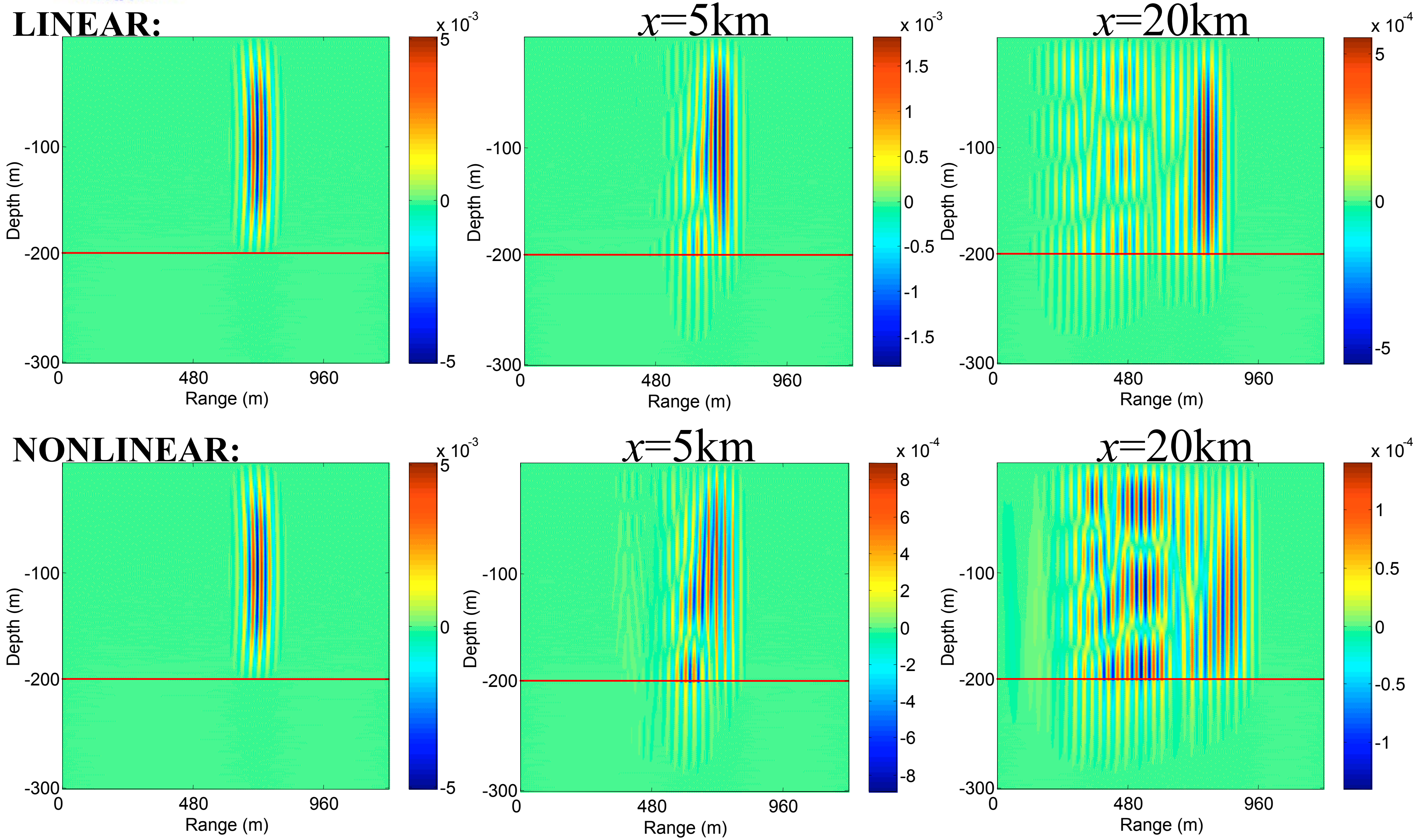
$$x_{\min} \le x_{NPE} \le x_{\max}$$

Advantage :
- thermoviscous absorption included (ultrasonics)

Advantage :
- refraction included (ocean waveguide)



4) Shallow water waveguide : snapshots

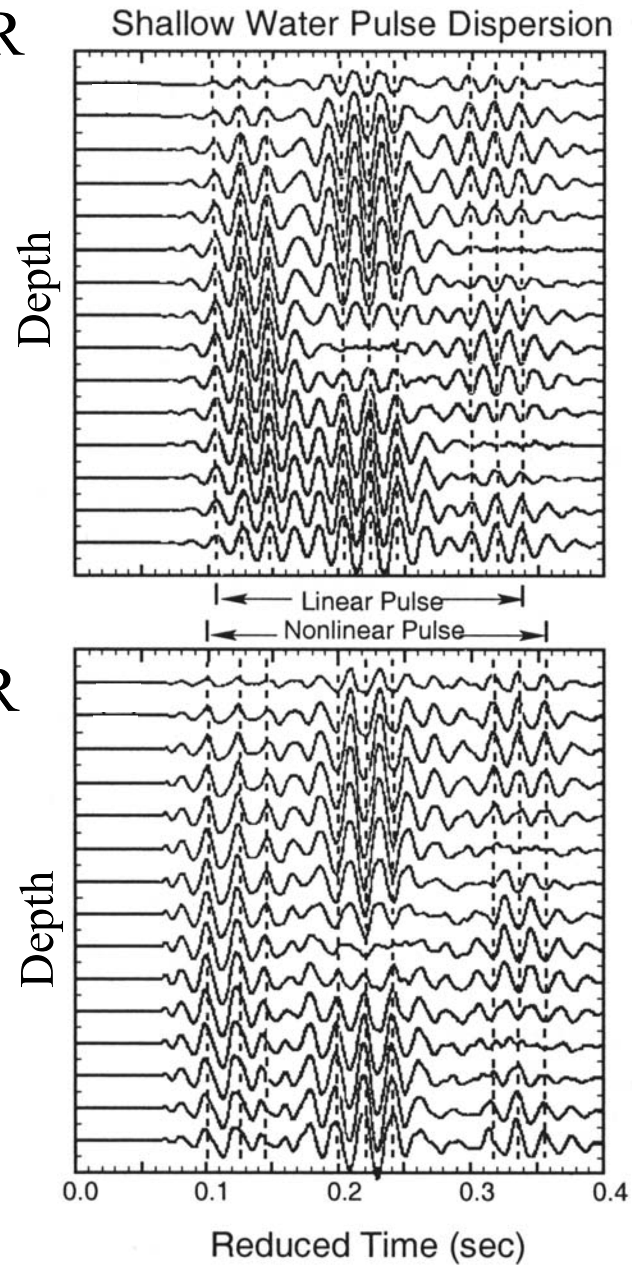


4) Shallow water waveguide : time series

LINEAR

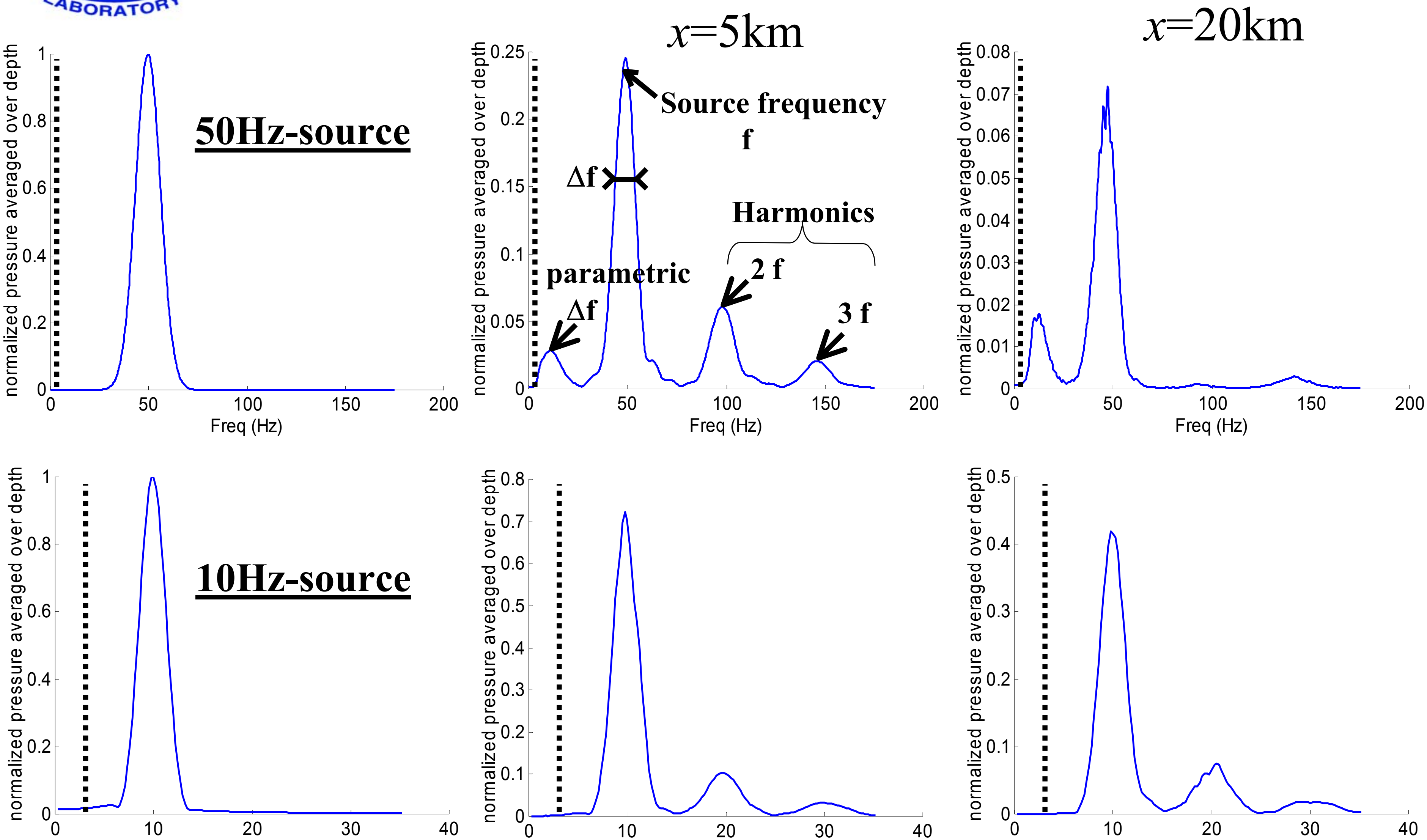
x=20km

NONLINEAR



- nonlinear steepening of the waveform
- larger signal duration

4) Shallow water waveguide : spectra



$f_c \approx 4$Hz

(200m shallow water waveguide cutoff frequency)

5) Conclusion

- KZK & NPE are consistent with each other

- <u>Main features of the NPE :</u>

- **developed for shock propagation in an ocean waveguide**
  → Refraction

- **accurate transport scheme**
  → Losses at shocks

- **thermoviscous dissipation can be easily added for high frequency problems**

$+\frac{\delta}{2}\partial_x^2 p$ on the Right Hand Side of the NPE [Too & Lee, *JASA*, **97**, 867-874, (1994)]

NPE Algorithm characteristics

- Numerical shemes

$$\underbrace{\partial_t p = -\partial_x\left[c_1 p + \frac{\beta}{2\rho_0 c_0} p^2\right]}_{\text{Refraction + Nonlinear steepening}} \underbrace{- \frac{c_0}{2}\int_{-\infty}^{x} \nabla_\perp^2 p \, dx}_{\text{Diffraction}}$$

Diffraction Step :

Crank-Nicholson scheme

Refraction + Nonlinear steepening Step :

Second order upwind flux corrected transport scheme

[B. E. McDonald, *J. Comp. Phys.* **56**, 448-460, (1984)]

⇨ Accounts properly for shock dissipation automatically

⇨ Avoids Gibbs 'oscillations

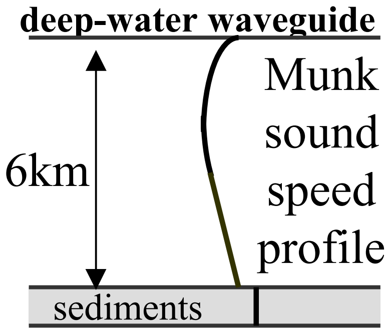
- Accounts for the absorption in the sediment layer of the ocean waveguide

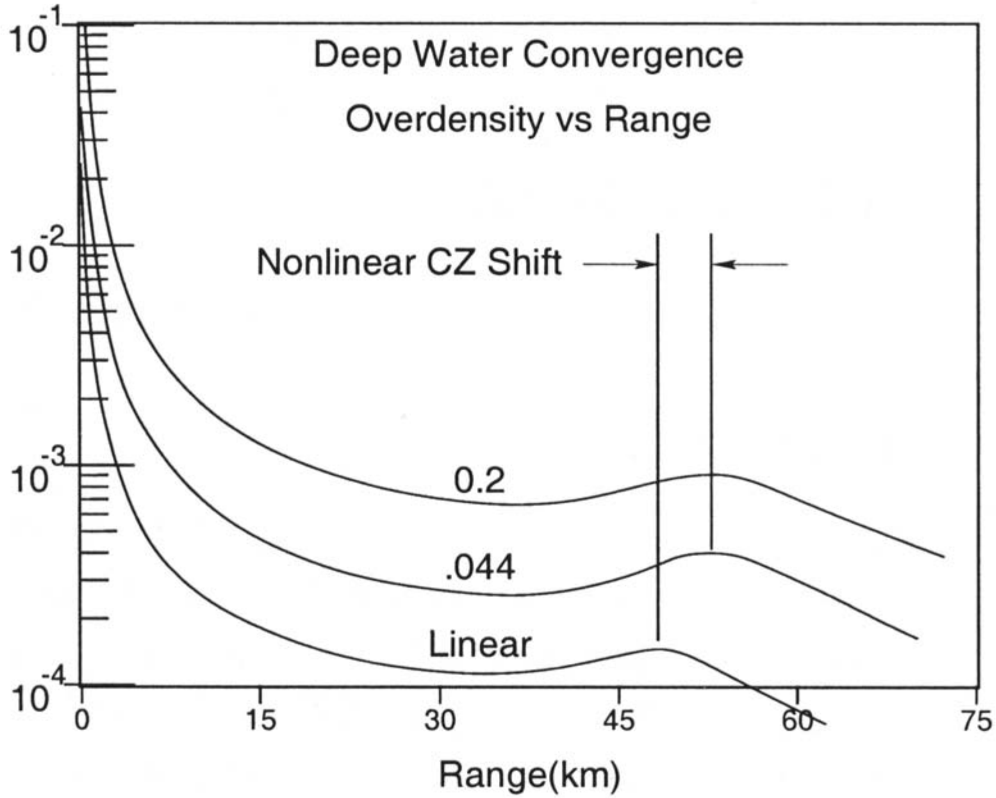
- Thermoviscous dissipation can be easily added for high frequency problems

$+\frac{\delta}{2}\partial_x^2 p$ on the Right Hand Side of the NPE [Too & Lee, *JASA*, **97**, 867-874, (1994)]

Deep water waveguide

Nonlinear effects → self-refraction





$\beta\, c_0 \frac{\rho'}{\rho_0} >> c_1$ ⇨ nonlinear waves focus at a greater range than the linear one