Equations of Nonlinear Acoustics and Weak Shock Propagation

Kaelig Castor*, B. Edward McDonald†, and William A. Kuperman*

*Scripps Institute of Oceanography, La Jolla, California 92093 USA
†US Naval Research Laboratory, Washington DC 20375 USA

Abstract The Nonlinear Progressive Wave Equation (NPE) was developed in the 1980's and 1990's as a time domain model for long range shock propagation [McDonald and Kuperman, J. Acoust. Soc. Am. 81, 1406, 1987]. Its form is quite different from the KZK equation [Zabolotskaya and Khokhlov, Sov. Phys.-Acoust. 15, 35, 1969]. We demonstrate that the NPE and the KZK equation are equivalent within their respective approximations. We give sample calculations with the NPE which demonstrate the motivation for its development. The examples illustrate self refraction and nonlinear dispersion in an ocean waveguide.

INTRODUCTION

During the 1980's the nonlinear progressive wave equation (NPE) theory[1] and numerics[2] were developed to obtain accurate and affordable simulations of shock propagation in the deep ocean out to convergence zone ranges (of order 30 - 50km). Researchers trying to solve the problem with hydrocodes found poor results when the shock became weak. The mismatch between wave propagation speed and the speed of the fluid itself lowers hydrocode accuracy and efficiency. In addition, the physics of refraction in the ocean demands an accurate accounting of small gradients in sound speed.

The same physics problem had been faced in the 1970's during studies of sonic boom effects from supersonic aircraft[3]. During that investigation solutions were attempted by applying one dimensional weak shock theory within ray tubes. Such solutions fail at caustics and focal points, where one would be most interested in peak amplitudes. The NPE has been successfully used to estimate physics of focusing weak shocks for sonic booms in the atmosphere[4] and for explosions in the ocean[5,6].

THEORY

The NPE arises from the Euler equations for inviscid fluid dynamics:

$$\frac{\partial \rho}{\partial t} = -\partial_i \rho v_i \quad \text{and} \quad \frac{\partial \rho v_i}{\partial t} = -\partial_j \rho v_i v_j - \partial_i p. \tag{1}$$

where ρ is density, v is velocity, and p is pressure. The Euler equations give without approximation a suggestive nonlinear wave equation

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p + \partial_i \partial_j (\rho v_i v_j) \tag{2}$$

The NPE is a nonlinear paraxial approximation derived[1] from eq. (2) by a series expansion in which the following variables and operators are of first order in the expansion parameter ϵ: wave amplitude, environmental variation, transverse Laplacian $\nabla_\perp^2 = (\partial_y^2 + \partial_z^2)$, and the wave-following time derivative $D/Dt = \partial/\partial_t + c_0 \partial/\partial_x$. In the resulting set of equations, order zero describes a piecewise constant medium at rest, order ϵ describes arbitrary linear plane waves $f(x - c_0 t)$, and order ϵ^2 is the NPE. The NPE for the acoustic overpressure p' is

$$\frac{Dp'}{Dt} = -\frac{\partial}{\partial x}\left(\underset{\text{(a)}}{c_1 p'} + \underset{\text{(b)}}{\frac{\beta}{2\rho_0 c_0} p'^2}\right) - \frac{c_0}{2}\int_\infty^x \underset{\text{(c)}}{(\partial_y^2} + \underset{\text{(d)}}{\partial_z^2)} p' dx \tag{3}$$

Subscript 0 indicates the constant reference value, and subscript 1 indicates departure of the environment from reference values so that the linear sound speed is $c(x, y, z) = c_0 + c_1(x, y, z)$. The primary propagation direction is taken to be x. For ocean or atmospheric problems z is upward, and y is the horizontal direction normal to x. The terms on the right hand side of (3) represent (a) refraction, (b) nonlinearty, (c) geometric spreading (for axial symmetry), and (d) diffraction.

$$\beta = 1 + \frac{\rho}{c}\frac{\partial c}{\partial \rho}\bigg|_0$$

is the coefficient of nonlinearity evaluated adiabatically in the undisturbed medium. An adiabatic equation of state is valid since weak shock heating is *third* order in amplitude[7].

For irrotational motion the KZK equation[8,9] for the velocity potential $\mathbf{v} = -\nabla\phi$ in the absence of attenuation and refraction may be written

$$\frac{\partial^2 \phi}{\partial t^2} - c_0^2 \nabla^2 \phi = \frac{\partial}{\partial t}\left[(\nabla\phi)^2 + (\beta - 1)c_0^{-2}(\frac{\partial \phi}{\partial t})^2\right] \tag{4}$$

Taking $\mathbf{v} = (u, v, w)$, we may write $\phi = -\int_\infty^x u dx$, so eq. (4) becomes

$$-\frac{\partial^2}{\partial t^2}\int_\infty^x u dx + c_0^2 \nabla^2 \int_\infty^x u dx = \frac{\partial}{\partial t}\left[\mathbf{v}^2 + (\beta - 1)c_0^{-2}\left(\frac{\partial}{\partial t}\int_\infty^x u dx\right)^2\right] \tag{5}$$

We replace $\partial/\partial t$ with $D/Dt - c_0 \partial/\partial x$. Then since the right side of (5) is of second order in wave amplitude, and since D/Dt is of first order in ϵ we may drop D/Dt terms on the right side to obtain to second order in ϵ

$$2c_0 \frac{D}{Dt}u - c_0^2 \frac{\partial}{\partial x}u + c_0^2 \nabla^2 \int_\infty^x u dx = -c_0 \frac{\partial}{\partial x}\left[\mathbf{v}^2 + (\beta - 1)u^2\right] \tag{6}$$

The second term on the left side of (6) cancels the part of the Laplacian term. On the right side we replace $\mathbf{v}^2$ with u^2 since transverse velocity components are of higher order than u. The result is

$$\frac{Du}{Dt} = -\frac{\partial}{\partial x}\left(\frac{\beta}{2}u^2\right) - \frac{c_0}{2}\int_\infty^x (\partial_y^2 + \partial_z^2) u dx \tag{7}$$

We may substitute $u = p'/\rho_0 c_0 + O(\epsilon)$ into (7) and find the equation to second order in ϵ,

$$\frac{Dp'}{Dt} = -\frac{\partial}{\partial x}\left(\frac{\beta}{2\rho_0 c_0}p'^2\right) - \frac{c_0}{2}\int_\infty^x (\partial_y^2 + \partial_z^2) p' dx, \tag{8}$$

which agrees with (3) when refraction is absent. Thus to their respective approximations the NPE and KZK are in agreement.

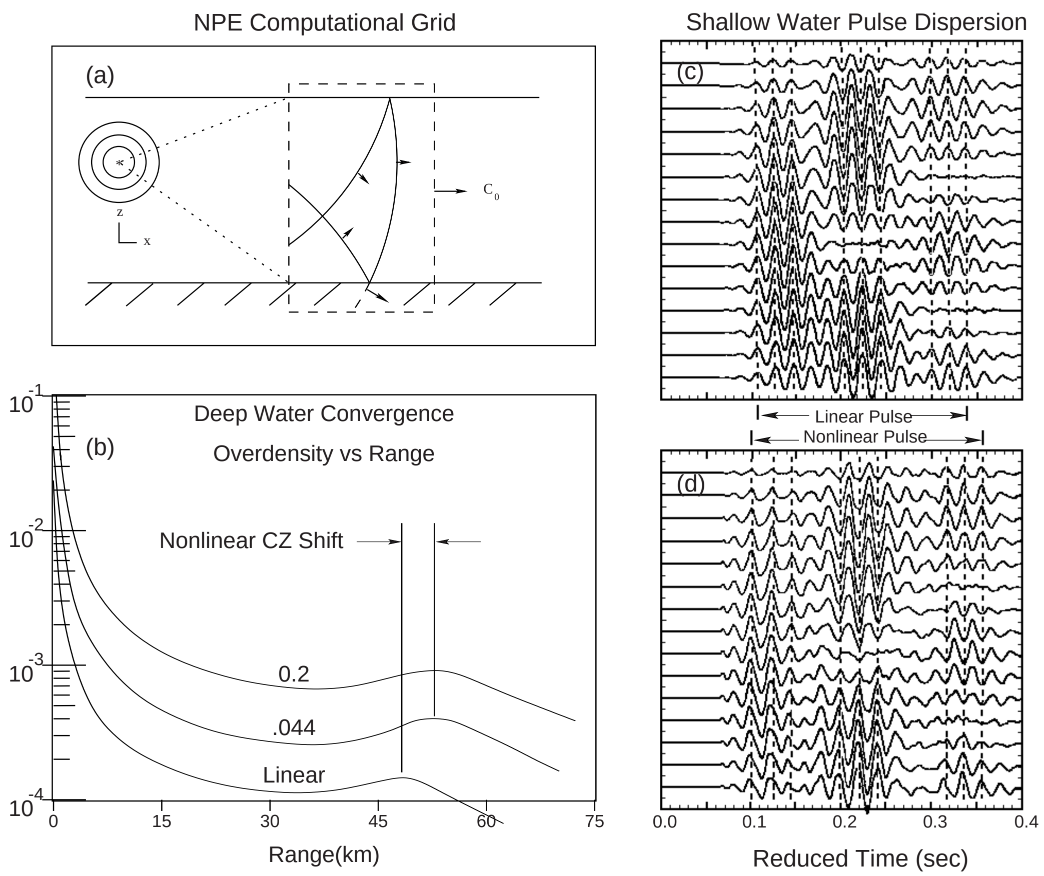

Figure 1. (a) The NPE wave- following grid (dashed box). (b) Peak overdensity vs range for linear and nonlinear convergence in a deep underwater sound channel. (c) Modal dispersion at 20km range from a linear source. (d) Mode- like dispersion at 20km range from a nonlinear source.

ILLUSTRATIVE RESULTS

Figure 1(a) shows the NPE's computation grid moving at speed c_0 away from the source (concentric circles). Fig. 1(b) gives an example of nonlinear defocusing of a shock wave propagating horizontally in an idealized sound channel with an idealized sound speed profile[6]. (This is also a form of self refraction). For this case the NPE grid was 1.05km wide and 6km deep. The curves represent peak overdensity vs range, and are labeled with the wave's peak overdensity at time zero. The nonlinear waves focus at a greater range than the linear one for the reason that while the wave has high amplitude, the nonlinear perturbation $\beta c_0 \rho'/\rho_0$ of the sound speed overwhelms the environmental gradient.

Figures 1(c) and (d) compare linear and nonlinear pulse propagation in a shallow water waveguide[5]. Both cases begin as a spherical wave consisting of five cycles of a 50Hz sine wave under a Gaussian envelope. The linear case 1(c) captures the proper separation of the first three ocean acoustic modes after propagation out to 20km range. The nonlinear case 1(d) shows wavefront steepening, plus increased wavepacket spreading. Both are due to the wave's nonlinear alteration of the background sound speed.

CONCLUDING REMARKS

We have shown that the NPE and KZK are consistent with each other, given their individual assumptions. NPE was originally formulated as an inviscid model for two reasons. First, on scales of interest to long range shock propagation the thickness of the shock front is not relevant. The internal constraints of mass conservation and monotonicity are sufficient to give proper shock jump conditions without having them imposed separately. Second, since NPE was developed during a time when hydrocodes were using artificial viscosity to control Gibbs' oscillations and numerical instabilities, it was desirable to show that such artificial viscosity was not needed. For application to small scale problems, the addition of molecular viscosity to NPE is quite easy, resulting in an extra term on the right hand side of eq. (3). Since NPE has been derived without the assumption of irrotationality it was quite easy to use for grazing incidence of a weak shock onto a free slip interface[10].

ACKNOWLEDGMENTS

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